(a). Suppose the eigenvalue and eigenvector are λ and v. $(v \neq 0)$.

Then: $Av = \lambda v$, which is equal to: $(A-\lambda I)\cdot v = 0$.

Since $v \neq 0$, we have: $\det(A-\lambda I) = 0$. That is: $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$.

$-\lambda^3 + 3\lambda + 2 = -(\lambda-2)\cdot(\lambda+1)^2 = 0$. So: $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$.

(b). $(A-\lambda I)\cdot v = 0$.

1°. $\lambda_1 = \lambda_2 = -1$. We have: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So: $v_a + v_b + v_c = 0$.

Since $v = \begin{pmatrix} v_a \\ v_b \\ -(v_a + v_b) \end{pmatrix}$, So: $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

2°. $\lambda_3 = 2$, we have: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So: $v_a = v_b = v_c$.

Since $v = \begin{pmatrix} v_a \\ v_a \\ v_a \end{pmatrix}$, so: $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c). $A = P\Lambda P^{-1}$.

Λ is a diagonal matrix comprosing of the eigenvalues of A.

so: $\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The colomns of P are eigenvectors of A, so: $P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

So: $A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$.

(d). By solving the matrix equation:

$$x = P\cdot\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 + c_3 \\ c_2 + c_3 \\ c_1 + c_3 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So we can get: $x = c_1 v_1 + c_2 v_2 + c_3 v_3$.

(e). $A = P\Lambda P^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$

$A^{100} = P\Lambda^{100}P^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$

So the eigenvalues of A^{100} are: $1, 1, 2^{100}$.

And the corresponding eigenvectors are: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

That is: $A^{100} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; $A^{100} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $A^{100} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2^{100} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{aligned} y = A^{100} \cdot x &= A^{100} \cdot (c_1 v_1 + c_2 v_2 + c_3 v_3) \\ &= c_1 \cdot A^{100} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot A^{100} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \cdot A^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= c_1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 2^{100} \cdot c_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

So: $\begin{cases} y_1 = -c_1 - c_2 + 2^{100} \cdot c_3 \\ y_2 = \quad\quad\; c_2 + 2^{100} \cdot c_3 \\ y_3 = c_1 \quad\quad + 2^{100} \cdot c_3 \end{cases}$

Since: $\begin{cases} x_1 = -c_1 - c_2 + c_3 \\ x_2 = \quad\quad\; c_2 + c_3 \\ x_3 = c_1 \quad\quad + c_3 \end{cases}$ and $\|y\|_2 = \|x\|_2$.

We can easily get $c_3 = 0$.

So, $x_1 + x_2 + x_3 = 0$.